Ordered Blue Schemes

Def: An ordered blueprinted space (OBlpr-space) is a pair $(X, \mathcal{O}_X)$, where:

- X is a topological space
- $\mathcal{O}_X$ is a sheaf of ordered blueprints on X

- For $x \in X$, the stalk in x is $\mathcal{O}_{X,x} := \operatorname{colim}_{x \in U} \mathcal{O}_X(U) \in \mathrm{OBl}_{pr}$

Rmk: OBl_{pr} is complete and cocomplete

- A morphism $\varphi^\#: \mathcal{A} \longrightarrow \mathcal{B}$ between sheaves in OBl_{pr} on X is local if the induced map $\varphi^\#_x: \mathcal{A}_x \longrightarrow \mathcal{B}_x$ satisfies $(\varphi^\#_x)^{-1}(\mathcal{B}_x^\times) \subseteq \mathcal{A}_x^\times$, $\forall x \in X$

(i.e., sends the maximal m-ideal of $\mathcal{A}_x$ to the maximal m-ideal of $\mathcal{B}_x$)

- A morphism of $\mathcal{OBl}_{pr}$-spaces is a pair $(\varphi, \varphi^{\#}) : X \longrightarrow Y$ where
  - $\varphi : X \longrightarrow Y$ is continuous
  - $\varphi^{\#} : \varphi^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is local

Note that $(\varphi^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,\varphi(x)}$

So one has the category $\mathcal{OBl}_{pr}\mathcal{S}p$ of ordered blueprinted spaces.

The Spectrum

Def: Let B be an ordered blueprint.
Define $\operatorname{Spec} B := \{m\text{-ideals of } B\}$
with topology generated by
$\{U_h := \{P \in \operatorname{Spec} B \mid h \notin P\} \mid h \in B^\bullet\}$

$\operatorname{Spec} B = \operatorname{Spec} B^\bullet$ as topological space

OBlpr context Mon_* context

And the sheaf $\mathcal{O}_{\operatorname{spec} B}$ generated by:

$$U_h \longmapsto B[h^{-1}] := \{h^n \mid n \in \mathbb{N}\}^{-1} B$$

Rmk: $\left(U_h, \mathcal{O}_{\operatorname{spec} B}|_{U_h}\right) \simeq \operatorname{Spec} B[h^{-1}]$

Remember:

$$\operatorname{Spec} S^{-1}B^{\cdot} \longleftrightarrow \{P \in \operatorname{Spec} B^{\cdot} \mid S \cap P = \emptyset\}$$

An affine ordered blue scheme is an OBlpr-space isomorphic to the spectrum of some ordered blueprint.

An ordered blue scheme is an OBlpr-space that has an open covering by affine ordered blue schemes.

$$X = \bigcup_{i \in I} X_i, \quad X_i \simeq \operatorname{Spec} B_i, \quad B_i \in \mathrm{OBlpr}, \ \forall i \in I$$

So one has the category OBSch of ordered blue schemes as the full subcategory of OBlprSp generated by these objects.

- One can extend the Spec construction to obtain a functor

$$\mathrm{Spec} : \mathrm{OBlpr}^{op} \longrightarrow \mathrm{OBSch} \qquad :$$

For $f: B \to C$ in OBl_{pr},

Let
$$\begin{aligned}\tilde{f}: \operatorname{Spec} C &\longrightarrow \operatorname{Spec} B \\ P &\longmapsto f^{-1}(P) \cap B^{\bullet}\end{aligned}$$

And $\tilde{f}^{\#}: \tilde{f}^{-1}\mathcal{O}_{\operatorname{Spec} B} \longrightarrow \mathcal{O}_{\operatorname{Spec} C}$ induced by

$$B[h^{-1}] = \mathcal{O}_{\operatorname{Spec} B}(U_{B,h}) \longrightarrow \tilde{f}_* \mathcal{O}_{\operatorname{Spec} C}(U_{B,h}) = C[f(h)^{-1}]$$

$(\tilde{f}^{-1} \dashv \tilde{f}_*)$

Defining $\Gamma : \mathrm{OBSch} \longrightarrow \mathrm{OBlpr}^{op}$

$$X \longmapsto \mathcal{O}_X(X)$$

$$\varphi : X \longrightarrow Y \longmapsto \mathcal{O}_Y(Y)$$

$$\begin{tikzcd}
\mathcal{O}_Y(Y) \arrow[d] \arrow[dr, "\Gamma(\varphi)"] & \\
\varphi^{-1}\mathcal{O}_Y(X) \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{O}_X(X)
\end{tikzcd}$$

One has a functor that is left adjoint to Spec and satisfies $\Gamma \circ \mathrm{Spec} = \mathrm{Id}_{\mathrm{OBlpr}^{op}}$

Examples: 1) For $\mathbb{F}_1 := \underline{1} = (\{0,1\}, \mathbb{N}, =)$,

$\operatorname{Spec} \mathbb{F}_1 = \{\{0\}\}$, $\mathcal{O}_{\operatorname{Spec} \mathbb{F}_1}(\{\{0\}\}) = \mathbb{F}_1$

is final in OBSch

2) For $B = \underline{1}[X,Y]$,

$$\begin{array}{ccccc}
\operatorname{Spec} B = \{ & \{0\}, & \langle X\rangle, \langle Y\rangle, & \langle X,Y\rangle & \} \\
 & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & \\
U_Y = & \{\{0\}, \langle X\rangle\} & & \{\{0\}, \langle Y\rangle\} & = U_X \\
 & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & \\
 & & \{\{0\}\} & &
\end{array}$$

3) Let B be any ordered blueprint. Then there exists

$$\mathbb{P}^1_B = \operatorname{colim}\left(\begin{tikzcd} \operatorname{Spec} B[X_0^{\pm 1}] \arrow[r, "\sim"] \arrow[d, phantom, sloped, "\subseteq"] & \operatorname{Spec} B[X_1^{\mp 1}] \arrow[d, phantom, sloped, "\subseteq"] \\ \operatorname{Spec} B[X_0] & \operatorname{Spec} B[X_1] \end{tikzcd}\right) \qquad X_0 \longmapsto X_1^{-1}$$

all morphisms in $\mathrm{OBSch}/_{\operatorname{Spec} B}$

$\Rightarrow \mathbb{P}^1_B$ comes equipped with a morph. $\mathbb{P}^1_B \to \operatorname{Spec} B$

Monoid Schemes $\hookrightarrow$ OBSch

Let $\mathcal{F} : \mathrm{Mon}_* \longrightarrow \mathrm{OBl}_{pr}$

$$M \longmapsto (M, \mathbb{N}[M \setminus \{0\}], =)$$

Let X be a monoid scheme

and $\mathcal{U} = \{ U_i = \mathrm{MSpec}\, M_i \}_{i \in I}$

the biggest affine open covering of X

Defining $\mathfrak{F}(X) := \underset{(\text{in Top})}{\operatorname{colim}}\, U_i \quad (= X)$

and $\mathcal{O}_{\mathfrak{F}(X)}(V) := \lim_{U_i \subseteq V} \mathfrak{F}(M_i)$
(in OBl_{pr})

one has

$$\mathfrak{F}(X) = \underset{(\text{in OBSch})}{\operatorname{colim}}\, \operatorname{Spec} \mathfrak{F} M_i$$

This $\mathfrak{F}$ extends to a fully faithful functor $\overline{\mathfrak{F}}$: Monoid Schemes $\longrightarrow$ OBSch s.t. if $g : \operatorname{MSpec} M \longrightarrow \operatorname{MSpec} A$ is induced by $\tilde{g} : A \longrightarrow M$, then

$$\overline{\mathfrak{F}}(g) = \operatorname{Spec} \mathfrak{F}(\tilde{g})$$

The idea of why this works is that $\mathfrak{F}$ preserves limits and $\overline{\mathfrak{F}}$ is the equality on the underlying top. spc

There is a functor on the other direction:

$$\mathcal{F}^{\bullet}: \mathrm{OBSch} \longrightarrow \text{Monoid Schemes}$$

$$(X, \mathcal{O}_X) \longmapsto (X, \mathcal{O}_X^{\bullet}), \text{ where}$$

$$\mathcal{O}_X^{\bullet}: \underset{\substack{\text{in} \\ X}}{U} \longmapsto \mathcal{O}_X(U)^{\bullet} \leftarrow \text{the monoid of } \mathcal{O}_X(U)$$

Example

4) If $B = \mathfrak{J}M$ for some $M \in \mathrm{Mon}_*$, then $\mathbb{P}^1_B = \tilde{\mathfrak{J}}(\mathbb{P}^1_M)$

and $\tilde{\mathfrak{J}}(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{P}^1_{\mathbb{C}}$.

Relation with semiring schemes

Let $X \in \mathrm{OBSch}$ and $\mathcal{U} = \{U_i = \operatorname{Spec} B_i\}_{i \in I}$ its biggest open affine covering (category s.t. $\operatorname{ob}\mathcal{C} = I$

$D : \mathcal{C} \longrightarrow \mathrm{OBSch}$)

Then the diagram D on the set I that describes X as $\operatorname{colim}_{i \in I} U_i$

Can be composed with the functor $\mathrm{Spec} \circ \tilde{\mathcal{G}}^{op} \circ \Gamma$ to obtain

Sring context

OBlpr context

a diagram $D_2 : \mathcal{C} \longrightarrow Sch_{\mathbb{N}}$

s.t. $\exists \operatorname{colim} D_2 \in Sch_{\mathbb{N}}$

$$\mathcal{G}^+ X := \operatorname{colim} D_2$$

This construction can be extended to a functor $\mathcal{G}^+ : \mathrm{OBSch} \longrightarrow \mathrm{Sch}_{\mathbb{N}}$ s.t.:

Where
$\tilde{\mathcal{G}} : \mathrm{OBl}_{pr} \longrightarrow \mathrm{Sring}$
$B \longmapsto B^+$

$\mathcal{G} : \mathrm{Sring} \longrightarrow \mathrm{OBl}_{pr}$
$R \longmapsto (R^{\cdot}, R, =)$

$$\begin{tikzcd}
\mathrm{OBl}_{pr}^{op} \arrow[r, "\tilde{\mathcal{G}}^{op}"] \arrow[d, "\mathrm{Spec}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Sring}^{op} \arrow[d, "\mathrm{Spec}"] \\
\mathrm{OBSch} \arrow[r, "\mathcal{G}^+"'] & \mathrm{Sch}_{\mathbb{N}}
\end{tikzcd}$$

But there is no functor

$\overline{\mathcal{G}} : \mathrm{Sch}_{\mathbb{N}} \longrightarrow \mathrm{OBSch}$ that

preserves colimits and satisfies

$$\begin{tikzcd}
\mathrm{Sring}^{op} \arrow[r, "\mathcal{G}^{op}"] \arrow[d, "\mathrm{Spec}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{OBlpr}^{op} \arrow[d, "\mathrm{Spec}"] \\
\mathrm{Sch}_{\mathbb{N}} \arrow[r, "\overline{\mathcal{G}}"'] & \mathrm{OBSch}
\end{tikzcd}$$

Because if there is such $\bar{g}$, then:

- $\bar{g}(\mathrm{Spec}\,\mathbb{Z}) = \mathrm{Spec}\,(\mathbb{Z}^\bullet; \mathbb{Z}, =)$ has only one closed point $= \mathbb{Z}^\bullet \setminus \mathbb{Z}^\times$

- $$\mathrm{Spec}\,\mathbb{Z} = \operatorname{colim}\left(\begin{tikzcd} & \mathrm{Spec}\,\mathbb{Z}[1/6] \arrow[dl, bend right, "\varphi"'] \arrow[dr, bend left, "\psi"] & \\ \mathrm{Spec}\,\mathbb{Z}[1/2] & & \mathrm{Spec}\,\mathbb{Z}[1/3] \end{tikzcd} \right)$$

Let $P_n := \mathbb{Z}[1/n] \setminus \mathbb{Z}[1/n]^\times$ the unique closed point of $\mathrm{Spec}\left(\mathcal{G}(\mathbb{Z}[1/n])\right)$

- Note that $3/1 \in P_2$, but $3/1 \in \mathbb{Z}[1/6]^\times$

$\Rightarrow P_2 \notin \mathrm{Im}\ \varphi$

- Analogously, $P_3 \notin \mathrm{Im}\ \psi$

The underlying topological space of $\overline{\mathcal{G}}$ calculated on the colimit $\#$ is

$$\operatorname{Spec} \mathcal{G}(\mathbb{Z}[1/2]) \sqcup \operatorname{Spec} \mathcal{G}(\mathbb{Z}[1/3]) \Big/ \sim$$

where $P \sim Q$ if $\exists\, J \in \operatorname{Spec} \mathcal{G}(\mathbb{Z}[1/6])$

s.t. $P = \overline{\mathcal{G}}(\varphi)(J) = \mathbb{Z}[1/2]^{\bullet} \cap J$

$Q = \overline{\mathcal{G}}(\psi)(J) = \mathbb{Z}[1/3]^{\bullet} \cap J$

that has $[P_2] \neq [P_3]$ as closed points, Abs!